

# ON THE DERIVATIVES OF THE LEMPert FUNCTIONS

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ABSTRACT. We show that if the Kobayashi–Royden metric of a complex manifold is continuous and positive at a given point and any non-zero tangent vector, then the "derivatives" of the higher order Lempert functions exist and equal the respective Kobayashi metrics at the point. It is a generalization of a result by M. Kobayashi for taut manifolds.

## 1. INTRODUCTION AND RESULTS

Let  $\mathbb{D} \subset \mathbb{C}$  be the unit disc. Let  $M$  be an  $n$ -dimensional complex manifold. Recall first the definitions of the Lempert function  $\tilde{k}_M$  and the Kobayashi–Royden pseudometric  $\kappa_M$  of  $M$ :

$$\tilde{k}_M^*(z, w) = \inf\{|\alpha| : \exists f \in \mathcal{O}(\mathbb{D}, M) : f(0) = z, f(\alpha) = w\},$$

$$\tilde{k}_M = \tanh^{-1} \tilde{k}_M^*,$$

$$\kappa_M(z; X) = \inf\{|\alpha| : \exists f \in \mathcal{O}(\mathbb{D}, M) : f(0) = z, \alpha f_*(d/d\zeta) = X\},$$

where  $X$  is a complex tangent vector to  $M$  at  $z$ . Note that such an  $f$  always exists (cf. [12]; according to [2], page 49, this was already known by J. Globevnik).

The Kobayashi pseudodistance  $k_M$  can be defined as the largest pseudodistance bounded by  $\tilde{k}_M$ . Note that if  $k_M^{(m)}$  denotes the  $m$ -th Lempert function of  $M$ ,  $m \in \mathbb{N}$ , that is,

$$k_M^{(m)}(z, w) = \inf\left\{\sum_{j=1}^m \tilde{k}_M(z_{j-1}, z_j) : z_0, \dots, z_m \in M, z_0 = z, z_m = w\right\},$$

then

$$k_M(z, w) = k_M^{(\infty)} := \inf_m k_M^{(m)}(z, w).$$

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By a result of M.-Y. Pang (see [9]), the Kobayashi–Royden metric is the "derivative" of the Lempert function for taut domains in  $\mathbb{C}^n$ ; more precisely, if  $D \subset \mathbb{C}^n$  is a taut domain, then

$$\kappa_D(z; X) = \lim_{t \rightarrow 0} \frac{\tilde{k}_D(z, z + tX)}{t}.$$

In [6], S. Kobayashi introduces a new invariant pseudometric, called the Kobayashi–Buseman pseudometric in [3]. One of the equivalent ways to define the Kobayashi–Buseman pseudometric  $\hat{\kappa}_M$  of  $M$  is just to set  $\hat{\kappa}_M(z; \cdot)$  to be largest pseudonorm bounded by  $\kappa_M(z; \cdot)$ . Recall that

$$\hat{\kappa}_M(z; X) = \inf \left\{ \sum_{j=1}^m \kappa_M(z; X_j) : m \in \mathbb{N}, \sum_{j=1}^m X_j = X \right\}.$$

Thus it is natural to consider the new function  $\kappa_M^{(m)}$ ,  $m \in \mathbb{N}$ , namely,

$$\kappa_M^{(m)}(z; X) = \inf \left\{ \sum_{j=1}^m \kappa_M(z; X_j) : \sum_{j=1}^m X_j = X \right\}.$$

We call  $\kappa_M^{(m)}$  the *m-th Kobayashi pseudometric* of  $D$ . It is clear that  $\kappa_M^{(m)} \geq \kappa_M^{(m+1)}$  and if  $\kappa_M^{(m)}(z; \cdot) = \kappa_M^{(m+1)}(z; \cdot)$  for some  $m$ , then  $\kappa_M^{(m)}(z; \cdot) = \kappa_D^{(j)}(z; \cdot)$  for any  $j > m$ . It is shown in [8] that  $\kappa_M^{(2n-1)} = \kappa_M^{(\infty)} := \hat{\kappa}_M$ , and  $2n - 1$  is the optimal number, in general.

We point out that all the introduced objects are upper semicontinuous. Recall that this is true for  $\kappa_M$  (cf. [7]). It remains to check this for  $\tilde{k}_M$ . We shall use a standard reasoning. Fix  $r \in (0, 1)$  and  $z, w \in M$ . Let  $f \in \mathcal{O}(\mathbb{D}, M)$ ,  $f(0) = z$  and  $f(\alpha) = w$ . Then  $\tilde{f} = (f, \text{id}) : \Delta \rightarrow \tilde{M} = M \times \Delta$  is an embedding. Setting  $\tilde{f}_r(\zeta) = \tilde{f}(r\zeta)$ , by [10], Lemma 3, we may find a Stein neighborhood  $S \subset \tilde{M}$  of  $\tilde{f}_r(\mathbb{D})$ . Embed  $S$  as a closed complex manifold in some  $\mathbb{C}^N$  and denote by  $\psi$  the respective embedding. Moreover, there is an open neighborhood  $V \subset \mathbb{C}^N$  of  $\psi(S)$  and a holomorphic retraction  $\theta : V \rightarrow \psi(S)$ . Then, for  $z'$  near  $z$  and  $w'$  near  $w$ , we may find, as usual,  $g \in \mathcal{O}(\mathbb{D}, V)$  such that  $g(0) = \psi(z')$  and  $g(\alpha/r) = \psi(w', \alpha)$ . Denote by  $\pi$  the natural projection of  $\tilde{M}$  onto  $M$ . Then  $h = \pi \circ \psi^{-1} \circ \theta \circ g \in \mathcal{O}(\mathbb{D}, M)$ ,  $h(0) = z'$  and  $h(\alpha/r) = w'$ . So  $r\tilde{k}_M^*(z', w') \leq \alpha$ , which implies that  $\limsup_{z' \rightarrow z, w' \rightarrow w} \tilde{k}_M(z', w') \leq \tilde{k}_M(z, w)$ .

To extend Pang's result on manifolds, we have to define the "derivatives" of  $k_M^{(m)}$ ,  $m \in \mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ . Let  $(U, \varphi)$  be a holomorphic chart

near  $z$ . Set

$$\mathcal{D}k_M^{(m)}(z; X) = \limsup_{t \rightarrow 0, w \rightarrow z, Y \rightarrow \varphi_* X} \frac{k_M^{(m)}(w, \varphi^{-1}(\varphi(w) + tY))}{|t|}.$$

Note that this notion does not depend on the chart used in the definition and

$$\mathcal{D}k_M^{(m)}(z; \lambda X) = |\lambda| \mathcal{D}k_M^{(m)}(z; X), \quad \lambda \in \mathbb{C}.$$

Replacing  $\limsup$  by  $\liminf$ , we define  $\underline{\mathcal{D}}k_M^{(m)}$ .

From M. Kobayashi's paper [5] it follows that, if  $M$  is a taut manifold, then

$$\hat{\kappa}_M(z; X) = \mathcal{D}k_M(z; X) = \underline{\mathcal{D}}k_M(z; X),$$

that is, the Kobayashi–Buseman metric is the "derivative" of the Kobayashi distance. The proof there also leads to

$$(*) \quad \kappa_M^{(m)}(z; X) = \mathcal{D}k_M^{(m)}(z; X) = \underline{\mathcal{D}}k_M^{(m)}(z; X), \quad m \in \mathbb{N}^*.$$

We say that a complex manifold  $M$  is hyperbolic at  $z$  if  $k_M(z, w) > 0$  for any  $w \neq z$ . We point out that the following conditions are equivalent:

- (i)  $M$  is hyperbolic at  $z$ ;
- (ii)  $\liminf_{z' \rightarrow z, w \in M \setminus U} \tilde{k}_M(z', w) > 0$  for any neighborhood  $U$  of  $z$ ;
- (iii)  $\underline{\kappa}_M(z; X) := \liminf_{z' \rightarrow z, X' \rightarrow X} \kappa_M(z'; X') > 0$  for any  $X \neq 0$ ;

The implication (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) are almost trivial (cf. [3]) and the implication (iii)  $\Rightarrow$  (i) is a consequence of the fact that  $k_M$  is the integrated form of  $\kappa_M$ .

In particular, if  $M$  is hyperbolic at  $z$ , then it is hyperbolic at any  $z'$  near  $z$ .

Since if  $M$  is taut, then it is  $k$ -hyperbolic and  $\kappa_M$  is a continuous function, the following theorem is a generalization of (\*).

**Theorem 1.** *Let  $M$  be a complex manifold and  $z \in M$ .*

- (i) *If  $M$  is hyperbolic at  $z$  and  $\kappa_M$  is continuous at  $(z, X)$ , then*

$$\kappa_M(z; X) = \mathcal{D}\tilde{k}_M(z; X) = \underline{\mathcal{D}}\tilde{k}_M(z; X).$$

- (ii) *If  $\kappa_M$  is continuous and positive at  $(z, X)$  for any  $X \neq 0$ , then*

$$\kappa_M^{(m)}(z; \cdot) = \mathcal{D}k_M^{(m)}(z; \cdot) = \underline{\mathcal{D}}k_M^{(m)}(z; \cdot), \quad m \in \mathbb{N}^*.$$

The first step in the proof of Theorem 1 is the following

**Proposition 2.** *For any complex manifold  $M$  one has that*

$$\kappa_M^{(m)} \geq \mathcal{D}k_M^{(m)}, \quad m \in \mathbb{N}^*.$$

Note that when  $M$  is a domain, a weaker version of Proposition 2 can be found in [3], namely,  $\hat{\kappa}_M \geq \mathcal{D}k_M$  (the proof is based on the fact that  $\mathcal{D}k_M(z; \cdot)$  is a pseudonorm).

## 2. EXAMPLES

The following examples show that the assumption on continuity in Theorem 1 is essential.

- Let  $A$  be a countable dense subset of  $\mathbb{C}_*$ . In [1] (see also [3]), a pseudoconvex domain  $D$  in  $\mathbb{C}^2$  is constructed such that:

(a)  $(\mathbb{C} \times \{0\}) \cup (A \times \mathbb{C}) \subset D$ ;

(b) if  $z_0 = (0, t) \in D$ ,  $t \neq 0$ , then  $\kappa_D(z_0; X) \geq C\|X\|$  for some  $C = C_t > 0$ . (One can be shown that even  $\mathcal{D}k_D(z_0; X) \geq C\|X\|$ .)

Then it is easy to see that  $\underline{\kappa}_D(\cdot; e_2) = \mathcal{D}k_D^{(3)}(\cdot; e_2) = k_D^{(5)} = 0$  and  $\hat{\kappa}_D(z_0; X) \geq c\|X\|$ , where  $e_2 = (0, 1)$  and  $c > 0$ . Thus

$$\hat{\kappa}_D(z_0; X) > \underline{\kappa}_D(z_0; e_2) = \mathcal{D}k_D^{(3)}(z_0; e_2) = \mathcal{D}k_D^{(5)}(z_0; X), \quad X \neq 0.$$

This phenomena obviously extends to  $\mathbb{C}^n$ ,  $n > 2$  (by considering  $D \times \mathbb{D}^{n-2}$ ). So the inequalities in Proposition 2 are strict in general.

- If  $D$  is a pseudoconvex balanced domain with Minkowski function  $h_D$ , then (cf. [3])

$$h_D = \kappa_D(0; \cdot) = \mathcal{D}\tilde{k}_D(0; \cdot).$$

Therefore,  $\mathcal{D}\tilde{k}_D(0; X) > \underline{\mathcal{D}\tilde{k}_D}(0; X)$  if  $\kappa_D(0; \cdot)$  is not continuous at  $X$ . On the other hand, if  $\hat{D}$  denotes the convex hull of  $D$ , then

$$h_{\hat{D}} = \hat{\kappa}_D(0; \cdot) = \mathcal{D}k_D(0; \cdot) = \underline{\mathcal{D}k}_D(0; \cdot) = \underline{\kappa}_D(0; \cdot).$$

- Modifying the first example leads to a pseudoconvex domain  $D \subset \mathbb{C}^2$  with

$$L_{\mathcal{D}k_D}(\gamma) > 0 = L_{k_D}(\gamma) = L_{\underline{\mathcal{D}k}_D}(\gamma),$$

where  $\gamma : [0, 1] \rightarrow \mathbb{C}^2$ ,  $\gamma(t) := (ti/2, 1/2)$ , and  $L_{\bullet}(\gamma)$  denotes the respective length.

Indeed, choose a dense sequence  $(r_j)$  in  $[0, i/2]$ . Put

$$u(\lambda) = \sum_{k=1}^{\infty} \frac{1}{k^2} \log \frac{|\lambda - 1/k|}{4}, \quad v(\lambda) = \sum_{j=1}^{\infty} \frac{u(\lambda/2 - r_j)}{2j^2}, \quad \lambda \in \mathbb{C},$$

and

$$D = \{z \in \mathbb{C}^2 : \psi(z) = |z_2|e^{\|z\|^2 + v(z_1)} < 1\}.$$

It is easy to see that  $v$  is a subharmonic function on  $\mathbb{C}$ . Hence  $D$  is a pseudoconvex domain with  $(\mathbb{C} \times \{0\}) \cup (\bigcup_{j,k=1}^{\infty} \{r_j + 1/k\} \times \mathbb{C}) \subset D$ .

Observe that  $u|_{\mathbb{D}} < -1$  and so  $D$  contains the unit ball  $\mathbb{B}_2$ . Note also that

$$k_D(a, b) = 0, \quad a, b \in \gamma([0, 1]).$$

Set  $\hat{\psi}(z) = \|z\|^2/2 - \log \psi(z)$ . Fix  $z^0 \in \mathbb{B}_2$  with  $\operatorname{Re} z_1^0 \leq 0$ ,  $\operatorname{Im} z_2^0 \geq 1/e$ . Since  $u(\lambda) \geq u(0)$  for  $\operatorname{Re} \lambda \leq 0$ , we have

$$\|z^0\|/2 < \hat{\psi}(z^0) < 1 - u(0) =: 8C.$$

Let  $\varphi \in \mathcal{O}(\mathbb{D}, D)$ ,  $\varphi(0) = z^0$ . Following the estimates in the proof of Example 3.5.10 in [3], we see that  $\|\varphi'(0)\| < C$ . Hence,  $\kappa_D(z^0; X) \geq C\|X\|$ ,  $X \in \mathbb{C}^2$ . Since  $k_D$  is the integrated form of  $\kappa_D$ , it follows that

$$k_D(a, a - te_1) \geq Ct, \quad a \in \gamma([0, 1]), \quad 0 \leq t \leq 1/2 - 1/e, \quad e_1 = (1, 0).$$

Hence  $\mathcal{D}k_D(a; e_1) \geq C$  and therefore,  $L_{\mathcal{D}k_D}(\gamma) \geq C/2 > 0$ , which completes the proof of this example.

Note that it shows that, with respect to the lengths of curves,  $\mathcal{D}k_D$  behaves different than the "real" derivative of  $k_D$  (cf. [11] or [4], page 12). Moreover, it implies that, in general,  $\mathcal{D}k_D \neq \underline{\mathcal{D}}k_D$ .

**Questions.** It will be interesting to know examples showing that, in general,  $\kappa_D \neq \tilde{\mathcal{D}}k_D$ . It remains also unclear whether  $\mathcal{D}k_D$  is holomorphically contractible (see [3]). Recall that  $\int \mathcal{D}k_D = k_D$ ; but we do not know if  $\int \underline{\mathcal{D}}k_D = k_D$ .

### 3. PROOFS

*Proof of Proposition 2.* First, we shall consider the case  $m = 1$ . The key is the following

**Theorem 3.** \* [10] *Let  $M$  be an  $n$ -dimensional complex manifold and  $f \in \mathcal{O}(\mathbb{D}, M)$  regular at 0. Let  $r \in (0, 1)$  and  $D_r = r\mathbb{D} \times \mathbb{D}^{n-1}$ . Then there exists  $F \in \mathcal{O}(D_r, M)$ , which is regular at 0 and  $F|_{r\mathbb{D} \times \{0\}} = f$ .*

Since  $\kappa_M(z; 0) = \tilde{\mathcal{D}}k_M(z; 0) = 0$ , we may assume that  $X \neq 0$ . Let  $\alpha > 0$  and  $f \in \mathcal{O}(\mathbb{D}, M)$  be such that  $f(0) = z$  and  $\alpha f_*(d/d\zeta) = X$ . Let  $r \in (0, 1)$  and  $F$  as in Theorem 3. Since  $F$  is regular at 0, there exist open neighborhoods  $U = U(z) \subset M$  and  $V = V(0) \subset D_r$  such that  $F|_V : V \rightarrow U$  is biholomorphic. Hence  $(U, \varphi)$  with  $\varphi = (F|_V)^{-1}$ , is a chart near  $z$ . Note that  $\varphi_*(X) = \alpha e_1$ , where  $e_1 = (1, 0, \dots, 0)$ .

If  $w$  and  $Y$  are sufficiently near  $z$  and  $\alpha e_1$ , respectively, then

$$g(\zeta) := F(\varphi(w) + \zeta Y/\alpha), \quad \zeta \in r^2\mathbb{D},$$

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\*We may replace Theorem 3 by the approach used in the proof of the upper semicontinuity of  $\tilde{k}_M$ .

belongs to  $\mathcal{O}(r^2\mathbb{D}, M)$  with  $g(0) = w$  and  $g(t\alpha) = \varphi^{-1}(\varphi(w) + tY)$ ,  $t < r^2/\alpha$ . Therefore,  $r^2\tilde{k}_M^*(w, \varphi^{-1}(\varphi(w) + tY)) \leq t\alpha$ . Hence  $r^2\mathcal{D}\tilde{k}_M(z; X) \leq \alpha$ . Letting  $r \rightarrow 1$  and  $\alpha \rightarrow \kappa_M(z; X)$  we get that  $\mathcal{D}\tilde{k}_M(z; X) \leq \kappa_M(z; X)$ .

Let now  $m \in \mathbb{N}$ . By definition,  $\kappa_M^{(m)}(z; \cdot)$  is the largest function with the following property:

For any  $X = \sum_{j=1}^m X_j$  one has that  $\kappa_M^{(m)}(z; X) \leq \sum_{j=1}^m \kappa_M(z; X_j)$ .

To prove that  $\kappa_M^{(m)} \geq \mathcal{D}k_M^{(m)}$  it suffices to check that  $\mathcal{D}k_M^{(m)}(z; \cdot)$  has the same property. Following the above notation and choosing  $Y_j \rightarrow \varphi_* X_j$  with  $\sum_{j=1}^m Y_j = Y$ , we set  $w_0 = w$  and  $w_j = \varphi^{-1}(\varphi(w) + t \sum_{k=1}^j Y_k)$ . Since

$$k_M^{(m)}(w, w_q) \leq \sum_{j=1}^m \tilde{k}_M(w_{j-1}, w_j),$$

it follows by the case  $m = 1$  that

$$\mathcal{D}k_M^{(m)}(z; X) \leq \sum_{j=1}^m \mathcal{D}k_M(z; X_j) \leq \sum_{j=1}^m \kappa_M(z; X_j).$$

Finally, let  $m = \infty$  and  $n = \dim M$ . Since  $\hat{\kappa}_M = \kappa_M^{(2n-1)}$  and  $k_M \leq k_M^{(2n-1)}$ , we get that  $\mathcal{D}k_M \leq \hat{\kappa}_M$  using the case  $m = 2n - 1$ .  $\square$

*Proof of Theorem 1.* We may assume that  $X \neq 0$ . In virtue of Proposition 2, we have to show that

$$\kappa_M^{(m)}(z; X) \leq \underline{\mathcal{D}}k_M^{(m)}(z; X).$$

For simplicity we assume that  $M$  is a domain in  $\mathbb{C}^n$ .

(i) Fix a neighborhood  $U = U(z) \Subset M$ . Applying the hyperbolicity of  $M$  at  $z$ , there are a neighborhood  $V = V(z) \subset U$  and a  $\delta \in (0, 1)$  such that, if  $h \in \mathcal{O}(\mathbb{D}, M)$  with  $h(0) \in V$ , then  $h(\delta\mathbb{D}) \subset U$ . Hence, by the Cauchy inequalities,  $\|h^{(k)}(0)\| \leq c/\delta^k$ ,  $k \in \mathbb{N}$ .

Now choose sequences  $M \ni w_j \rightarrow z$ ,  $\mathbb{C}_* \ni t_j \rightarrow 0$ , and  $\mathbb{C}^n \ni Y_j \rightarrow X$  such that

$$\frac{\tilde{k}_M(w_j, w_j + t_j Y_j)}{|t_j|} \rightarrow \underline{\mathcal{D}}\tilde{k}_M(z; X).$$

There are holomorphic discs  $g_j \in \mathcal{O}(\mathbb{D}, M)$  and  $\beta_j \in (0, 1)$  with  $g_j(0) = w_j$ ,  $g_j(\beta_j) = w_j + t_j Y_j$ , and  $\beta_j \leq \tilde{k}_M^*(w_j, w_j + t_j Y_j) + |t_j|/j$ . Note that  $\tilde{k}_M^*(w_j, w_j + t_j Y_j) \leq c_1 \|t_j Y_j\| \leq c_2 |t_j|$ .

Write

$$w_j + t_j Y_j = g_j(\beta_j) = w_j + g_j'(0)\beta_j + h_j(\beta_j).$$

Then

$$\|h_j(\beta_j)\| \leq c \sum_{k=2}^{\infty} \left(\frac{\beta_j}{\delta}\right)^k \leq c_3 |\beta_j|^2 \leq c_4 |t_j|^2, \quad j \geq j_0.$$

Put  $\widehat{Y}_j = Y_j - h_j(\beta_j)/t_j$ . We have that  $g_j(0) = w_j$  and  $\beta_j g'_j(0)/t_j = \widehat{Y}_j \rightarrow X$ . Therefore,

$$\kappa_M(w_j; \widehat{Y}_j) \leq \frac{\beta_j}{|t_j|} \leq \frac{\widetilde{k}_M^*(z_j, w_j + t_j Y_j)}{|t_j|} + \frac{1}{j}.$$

Hence with  $j \rightarrow \infty$ , we get that  $\kappa_M(z; X) = \underline{\kappa}_M(z; X) \leq \underline{\mathcal{D}}\widetilde{k}_M(z; X)$ .

(ii) The proof of the case  $m \in \mathbb{N}$  is similar to the next one and we omit it. Now, we shall consider the case  $m = \infty$ .

Note first that our assumption implies that  $M$  is hyperbolic at  $z$  and, by the contrary,

$$\forall \varepsilon > 0 \exists \delta > 0 : \|w - z\| < \delta, \|Y - X\| < \delta \|X\|$$

$$(1) \quad \Rightarrow |\kappa_M(w; Y) - \kappa_M(z; X)| < \varepsilon \kappa_M(z; X).$$

Moreover, the proof of (i) shows that

$$(2) \quad \widetilde{k}_M(a, b) \geq \kappa_M(a; b - a + o(a, b)), \text{ where } \lim_{a, b \rightarrow z} \frac{o(a, b)}{\|a - b\|} = 0.$$

Choose now sequences  $M \ni w_j \rightarrow z$ ,  $\mathbb{C}_* \ni t_j \rightarrow 0$ , and  $\mathbb{C}^n \ni Y_j \rightarrow X$  such that

$$\frac{k_M(w_j, w_j + t_j Y_j)}{|t_j|} \rightarrow \underline{\mathcal{D}}k_M(z; X).$$

There are points  $w_{j,0} = w_j, \dots, w_{j,m_j} = w_j + t_j X_j$  in  $M$  such that

$$(3) \quad \sum_{k=1}^{m_j} \widetilde{k}_M(w_{j,k-1}, w_{j,k}) \leq k_M(w_j, w_j + t_j Y_j) + \frac{1}{j}.$$

Set  $w_{j,k} = w_j$  for  $k > m_j$ . Since

$$k_M(w_j, w_{j,l}) \leq \sum_{j=1}^l \widetilde{k}_M(w_{j,k-1}, w_{j,k}) \leq k_M(w_j, w_j + t_j Y_j) + \frac{1}{j} \leq c_2 |t_j| + \frac{1}{j},$$

then  $k_M(w_j, w_{j,l}) \rightarrow 0$  uniformly in  $l$ . Then the hyperbolicity of  $M$  at  $z$  implies that  $w_{j,l} \rightarrow z$  uniformly in  $l$ . Indeed, assuming the contrary and passing to a subsequence, we may suppose that  $w_{j,l_j} \notin U$  for some  $U = U(z)$ . Then

$$0 = \lim_{j \rightarrow \infty} k_M(w_j, w_{j,l}) \geq \liminf_{z' \rightarrow z, w \in M \setminus U} \widetilde{k}_M(z', w) > 0,$$

a contradiction.

Fix now  $R > 1$ . Then (1) implies that

$$\kappa_M(z; w_{j,k} - w_{j,k-1}) \leq R\kappa_M(w_{j,k}; w_{j,k} - w_{j,k-1} + o(w_{j,k}, w_{j,k-1})), \quad j \geq j(R).$$

It follows by this inequality, (2) and (3) that

$$\sum_{k=1}^{m_j} \kappa_M(z; w_{j,k} - w_{j,k-1}) \leq Rk_M(w_j, w_j + t_j Y_j) + \frac{R}{j}.$$

Since  $\hat{\kappa}_M(z; t_j Y_j)$  is bounded by the first sum, we obtain that

$$\hat{\kappa}_M(z; Y_j) \leq R \frac{k_M(w_j, w_j + t_j Y_j) + 1/j}{|t_j|}.$$

Note that  $\hat{\kappa}_M(z; \cdot)$  is a continuous function. Hence with  $j \rightarrow \infty$  and  $R \rightarrow 1$ , we get that  $\hat{\kappa}_M(z; X) \leq \underline{D}k_M(z; X)$ .  $\square$

**Remark.** It follows by the above proofs and a standard diagonal process that  $\underline{\kappa}_M(z; \cdot) = \underline{D}\tilde{k}(z; \cdot)$  if  $M$  is hyperbolic at  $z$ .

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